Characterization of All Nonlinear Stabilizing Controllers via Observer based Kernel Representations

Kenji Fujimoto and Toshiharu Sugie
Department of Applied Systems Science, Graduate School of Engineering, Kyoto University, Uji, Kyoto 611, Japan.
E-mail: fuji@robot.kuass.kyoto-u.ac.jp, sugie@robot.kuass.kyoto-u.ac.jp

Abstract
This paper is concerned with the characterization of all nonlinear stabilizing controllers. Observer based kernel representations are introduced to avoid the difficulty which occurs in constructing the state-space realizations of input-output operators. By using observer based kernel representations, the parametrization of all the stabilizing plant-controller pairs is derived which is equivalent to the state-space parametrization. In addition the relationship between the kernel representation approach and the existing state-space approach is clarified.

1 Introduction
Coprime factorization approach is widely used for linear systems analysis and synthesis. In the last decade, a lot of research has been done on nonlinear extension of coprime factorizations which contains the consistency with right coprime factorizations. Particularly in the parametrization of stabilizing controllers, this causes a problem in the state-space realization. In this paper, we derive a global parametrization of a class of stabilizing controllers, which contains all the locally stabilizing ones, by using the concept of detectable kernel representations. As a matter of fact, the parametrization obtained here is locally equivalent to the result in [2]. This result shows the relation between kernel representation approach and the existing state-space approach.

2 Definitions
This section gives the definitions used in the present paper, according to Paice et.al. [6]–[8]

2.1 Signal spaces and stability
In this paper, a signal space \( \mathcal{Z} \) is taken to be a vector space of functions from a time domain to a Euclidean vector space. The signal space is partitioned into two disjoint subsets, the stable subset \( \mathcal{Z}^s \) and the unstable subset \( \mathcal{Z}^u \). Also it is assumed that \( \mathcal{Z}^s \) is a vector space.
An operator \( \Sigma \) with an input signal space \( \mathcal{U} \), an output signal space \( \mathcal{Y} \) and an initial condition \( x^0 \in \mathcal{X}^s \) is denoted by \( \Sigma^{x^0} : \mathcal{U} \to \mathcal{Y} \). It is said to be stable (input-output stable) if the following holds for any initial condition.

\[
u \in \mathcal{U}^s \Rightarrow \Sigma^{x^0}(u) \in \mathcal{Y}^s\quad (1)
\]
An invertible operator is called unimodular when it is stable and has a stable inverse.

2.2 Kernel representations
In this section the kernel representations are introduced.

Definition 1: A kernel representation of the operator \( \Sigma^{x^0} : \mathcal{U} \to \mathcal{Y} \) is an operator \( R^{x^0}_{\Sigma} : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z} \) such
Kernel representations are generalization of left factorizations. The results derived in the sequel considers the case \( z \neq 0 \) and uses the pseudo inverse \( R_G^* \) to simplify. The existence of such an operator is defined by well-definedness.

**Definition 2**: A kernel representation \( R_G \) of \( g \) is said to be well-defined if for each \( z \in Z \) the mapping \( \Sigma_z : U \rightarrow Y \) exists, i.e.

\[
y = \Sigma_z^\circ (u) \iff R_G^z(u, y) = z
\]

**2.3 Closed loop stability**

\[
\begin{array}{ccc}
  e_1 & G & y' \\
  u' & K & y & e_2
\end{array}
\]

**Figure 1**: The feedback system \( \{G, K\} \)

We now consider the closed loop system depicted in Figure 1. This closed loop system, denoted by \( \{G, K\} \), is said to be well-posed if the signal \((u, u', y, y')\) is uniquely determined, and it is said to be stable if it is well-posed and \((u, u', y, y') \in U^* \times U'^* \times Y^* \times Y'^*\). Now, \( G, K \) is assumed to have the following kernel representations.

\[
R_G : U \times Y \rightarrow Z_G
\]

(4)

\[
R_K : Y \times U \rightarrow Z_K
\]

(5)

At first, we consider additive disturbances are zero, i.e. \((e_1, e_2) = 0\). The kernel representation \( R_GK : U \times Y \rightarrow Z_K \times Z_G \) of the closed loop system \( \{G, K\} \) can be defined as

\[
\begin{bmatrix}
  z_K \\
  z_G
\end{bmatrix}
= R_GK
\begin{bmatrix}
  u \\
  y
\end{bmatrix} := \begin{bmatrix}
  R_K(y, u) \\
  R_G(u, y)
\end{bmatrix}
\]

(6)

Condensed notations are defined as follows and so on.

\[
w := (u, y) \in U \times Y := W
\]

(7)

\[
z_{GK} := (z_K, z_G) \in Z_K \times Z_G := Z_{GK}
\]

(8)

\[
e_{12} := (e_1, e_2) \in E_1 \times E_2 := E_{12}
\]

(9)

\[
x_{GK}^0 := (x_G^0, x_K^0) \in X_G^0 \times X_K^0 := X_{GK}^0
\]

(10)

Now the well-posedness and the stability of the closed loop system with a kernel representation are defined.

**Definition 3**: A closed loop system \( \{G, K\} \) with a kernel representation \( R_{GK} \) is (strongly) well-posed if the closed loop system \( \{G_{2G}, K_{2K}\} \) is well-posed for \( \forall (x_{GK}^0, z_{GK}) \in X_{GK}^0 \times Z_{GK} \) and \( e_{12} = 0 \) (resp. for \( \forall e_{12} \in E_{12}^0 \)), where \( x_G^0 \) and \( x_K^0 \) are independent. Moreover, suppose \( R_{GK} \) is stable, then the system \( \{G, K\} \) with \( R_{GK} \) is (strongly) internally stable if the system \( \{G_{2G}, K_{2K}\} \) is stable for \( \forall (z_{GK}^0, x_{GK}) \in Z_{GK}^0 \times X_{GK}^0 \) and \( e_{12} = 0 \) (resp. for \( \forall e_{12} \in E_{12}^0 \)), where \( x_G^0 \) and \( x_K^0 \) are independent.

**2.4 State-space realizations**

When we consider the state-space results, we assume any operator \( \Sigma^* : U \rightarrow Y \) has the following state-space realization

\[
\begin{cases}
  \dot{x} = f(x, u) \\
y = h(x, u)
\end{cases}
\]

(11)

where \( f \) and \( h \) are sufficiently smooth functions satisfying \( f(0,0) = 0 \) and \( h(0,0) = 0 \).

**Definition 4**: An operator \( \Sigma^* : U \rightarrow Y \) with the state space realization (11) is said to be stable if it is input-output stable and the state of its realization is stable, e.g. asymptotically stable, or input-to-state stable [9]–[11].

**3 The parametrization of the stabilizing plant-controller pairs**

**3.1 Preliminaries**

Paice et al. had already given the following result as the parametrization of all the stabilizing plant-controller pairs via kernel representations.

**Theorem 1** [8]: Consider an internally stable system \( \{G, K\} \) with a kernel representation \( R_{GK} \), and systems \( S, Q \) with stable kernel representations \( R_S : Z_K \times Z_G \rightarrow Z_S \), \( R_Q : Z_G \times Z_K \rightarrow Z_Q \) respectively, giving \( G_S \) and \( K_Q \) with the following stable kernel representations.

\[
\left( \begin{array}{c}
  R_{GQ} \\
  R_{GS}
\end{array} \right) = R_{GSKQ} := R_{SQ} \circ R_{GK}
\]

(12)

Then the closed loop system \( \{G_S, K_Q\} \) with the kernel representation \( R_{GSKQ} \) will be internally stable if and only if the closed loop system \( \{S, Q\} \) with the kernel representation \( R_{SQ} \) is internally stable. Furthermore, given an internally stable system \( \{G^*, K^*\} \) with a kernel representation \( R_{G^*K^*} \), where \( R_{G^*} : U \times Y \rightarrow Z_G \), and \( R_{K^*} : Y \times U \rightarrow Z_K \), then there exists an internally stable system \( \{S^*, Q^*\} \) with a kernel representation \( R_{S^*Q^*} \), such that \( R_{S^*} : Z_K \times Z_G \rightarrow Z_S \), \( R_{Q^*} : Z_G \times Z_K \rightarrow Z_Q \), and \( G^* = G^*, K^* = K^* \).

In this theorem, the kernel representations \( R_{GS} \) and \( R_{KQ} \) are defined by (12) and they contain the same operator \( R_{GK} \). Hence the state-space representations of \( G_S \) and \( K_Q \) have to contain the same state of \( G_K \), and it is not convenient to use it in a practical situation. Actually, this result does not give the stabilizing plant-controller pairs in the meaning of Definition 3, because \( R_{GSKQ} \) is not necessarily unimodular for \( \forall (x_{GQ}^0, x_{KQ}^0) \in X_{GQ}^0 \times X_{KQ}^0 \) unless the initial condition \( (x_{GQ}^0, x_{KQ}^0) = (x_{GQ}^0, x_{KQ}^0, x_{QK}^0, x_{QQ}^0, x_{QK}^0, x_{QK}^0) \) satisfies \( x_{GQ}^0 \neq x_{GQ}^0 \) and \( x_{KQ}^0 \neq x_{KQ}^0 \), that is, \( x_{QK}^0 \) and \( x_{QK}^0 \) are dependent each other.

Most of the existing results on the parametrization of stabilizing controllers using the input-output approach have the similar difficulty. The present paper proposes to use observer based kernel representations to avoid such a difficulty. In the sequel, \( z_G = R_G(\cdot) \) and \( z_Q = R_Q(\cdot) \) denote the two operators which have the same realizations but with different initial conditions, i.e. \( R_G = R_G^z \) and \( R_Q = R_Q^z \) respectively.
3.2 Observer based kernel representations and the parametrization

For the purpose stated in the previous section, we introduce observer based kernel representations here. As the preparation of it, we consider the linear case.

If an operator $\Sigma$ is linear, then a stable kernel representation $R_G$ of $\Sigma$ can be made from its left coprime factorization. A left coprime factorization of $\Sigma$ is related to its state observer. Actually, such $R_G$ is a state observer not only for $\Sigma$ but also an estimator of the external signal $z$ to $\Sigma_c$. Furthermore, given an internally stable linear system $(G, K)$ with a kernel linear representation $R_{GK}$, then actually $R_{GK}$ is a state observer of the closed loop system $(G_{zG}, K_{zG})$, and is also an estimator of the external signal $z_{GK}$ to the system $(G_{zG}, K_{zG})$.

Now we define observer based kernel representations as detectable kernel representations which have the property as in the linear case.

**Definition 5:** A kernel representation $R_{GK}$ of a closed loop system $(G, K)$ is detectable if the following holds for $\forall (x_{GK}, x_{GK}^s, w) \in X_{GK} \times X_{GK}^s \times W$.

$$(R_{GK} - R_{GK})(w) \in Z_{GK}^s$$

(13)

In this definition, $R_{GK}$ is an estimator of the external signal $z_{GK}$ to the system $(G_{zG}, K_{zG})$ (or an observer of $(G_{zG}, K_{zG})$), i.e. $z_{GK}$ in the figure denotes the estimated signal of $z_{GK}$.

**Remark 1:** The detectability of $R_{GK}$ defined in Definition 5 is equivalent to the property that $(R_{GK} \circ R_{GK}^{-1}(z_{GK}) - z_{GK}) \in Z_{GK}^s$ holds for $\forall z_{GK} \in Z_{GK}$. This can be regarded as one generalization of double Bezout identity.

Assuming the detectability of $R_{GK}$, we can derive a parametrization of stabilizing plant-controller pairs.

**Theorem 2:** Consider a system $(G, K)$ with a kernel representation $R_{GK}$ and suppose it is internally stable and strongly detectable. Then the system $(G_{zG}, K_{zG})$ with a kernel representation $R_{GzKq}$ defined as follows is internally stable, where $R_{Gz} : U \times Y \rightarrow Z_s$ and $R_{Kq} : Y \times U \rightarrow Z$.

$R_{Gz} := R_S \circ R_{GK} \text{ s.t. } R_{Gz}$ is well-defined (14)

$R_{Kq} := R_Q \circ R_{GK} \text{ s.t. } R_{Kq}$ is well-defined (15)

where $R_S$ and $R_Q$ are kernel representations of systems $S$ and $Q$, i.e. $R_S : Z_K \times Z_G \rightarrow Z_S$ and $R_Q : Z_G \times Z_K \rightarrow Z_Q$, such that the closed loop system $(S, Q)$ with $R_{SQ}$ is strongly internally stable and that $(G_{zG}, K_{zG})$ with $R_{GzKq}$ is well-posed.

Proof is clear and is omitted. Unfortunately this theorem does not give all the pairs of internally stable $S$ and $Q$ but gives a sufficient condition for it. However the following section shows that the similar parametrization of strongly internally stable $S$ and $K$ gives all the stabilizing pairs.

**Remark 2:** By Theorem 1, the internal stability of $(S, Q)$ with $R_{SQ}$ is a necessary condition for the internal stability of $(G_{zG}, K_{zG})$ with $R_{GzKq}$.

3.3 The parametrization with additive disturbances

This section considers the parametrization of stabilizing plant-controller pairs with the additive disturbance $e_{12} \neq 0$. Now the detectability of the kernel representation $R_{GK}$ is extended for strong internal stability.

**Definition 6:** A kernel representation $R_{GK}$ of a closed loop system $(G, K)$ is strongly detectable if the following holds for $\forall (x_{GK}, x_{GK}^s, w) \in X_{GK} \times X_{GK}^s \times W$.

$e_{12} \in \mathcal{E}_{12}^s \Leftrightarrow (R_{GK}(w) - R_{GK}(w + e_{12})) \in Z_{GK}^s$

(16)

This definition implies $R_{GK}$ is an observer of the system $(G_{zG}, K_{zG})$, which works well when the input $w$ has a stable additive disturbance $e_{12}$. The concept of strong detectability gives us the following result.

**Theorem 3:** Consider a system $(G, K)$ with a kernel representation $R_{GK}$ and suppose it is strongly internally stable and strongly detectable. Then all the strongly internally stable systems $(G^*, K^*)$ with a kernel representation $R_{G^*K^*}$, where $R_{G^*} : U \times Y \rightarrow Z_s$, and $R_{K^*} : Y \times U \rightarrow Z_{K^*}$, are given by

$R_{G^*} = R_{Gz} := R_S \circ R_{GK} \text{ s.t. } R_{Gz}$ is well-defined (17)

$R_{K^*} = R_{Kq} := R_Q \circ R_{GK} \text{ s.t. } R_{Kq}$ is well-defined (18)

where $R_S$ and $R_Q$ are kernel representations of systems $S$ and $Q$, i.e. $R_S : Z_K \times Z_G \rightarrow Z_S = Z_s$ and $R_Q : Z_G \times Z_K \rightarrow Z_Q = Z_{K^*}$, such that the closed loop system $(S, Q)$ with $R_{SQ}$ is strongly internally stable and that $(G_{zG}, K_{zG})$ with $R_{GzKq}$ is strongly well-posed. Furthermore, the initial states $x_{G}^s, x_{K}^s, x_{G}^r, x_{K}^r, x_{S}^s$, and $x_{Q}^s$ are independent.

**proof:** The sufficiency of the parametrization, namely the strong internal stability of $R_{GzKq}$ defined by (17) and (18) is clear. Therefore what we have to prove is the necessity of the parametrization, namely the fact that all the stabilizing pair $R_{Gz}$ and $R_{Kq}$ can be expressed in the form (17) and (18), and the fact that the initial states of all the operators can be independent.

Necessity of the parametrization: Suppose a strongly internally stable pair $R_{Gz}$ and $R_{Kq}$ be given. Then $R_S$ and $R_Q$ satisfying (17) and (18) are given by

$R_S = R_{Gz} \circ R_{GK}^{-1}$

(19)

$R_Q = R_{Kq} \circ R_{GK}^{-1}$

(20)

Now suppose the system $(S, Q)$ with the kernel representation $R_{SQ}$ is not strongly internally stable and define new signals as follows.

$e_{GK} = (e_K, e_G) := (z_K - z_G, z_G - z_G)$

(21)

The signals $e_K$ and $e_G$ are considered as the additive disturbances to the system $(S_{zG}, Q_{zG})$ and the following relation holds because of the strong detectability of $R_{GK}$.

$e_{12} \in \mathcal{E}_{12}^s \Leftrightarrow e_{GK} \in \mathcal{E}_{GK}^s$

(22)

Then there exists $(z_{SQ}, e_{GK}) \in Z_{SQ}^s \times \mathcal{E}_{GK}^s$ such that $z_{GK} \in Z_{GK}^s$ and this implies the existence of
The system pairs strong internal stability of Theorem 3 gives the parametrization of the pairs of Corollary 1, the independence of the initial states of the setting which is also valid in state space setting. In these realizations, the state $x$ can choose stable signal $e_{GK}$ irrespectively of the initial states $x^0_{GK}$ and $x^0_{SK}$. This implies that $(S, Q)$ with $RSQ$ has to be strongly internally stable without the knowledge of the initial states of $RGK$ and $RGK$.

The last statement in Theorem 3, i.e. the initial states of all operators are independent, is very important and this fact allows us the following state space parametrization.

**Corollary 1**: Consider a system $(G, K)$ with the following kernel representation $R_{GK}$ and suppose it is strongly internally stable and strongly detectable:

$$R_G : \begin{cases}
\dot{x}_G = f_G(x_G, u, y) \\
\dot{z}_G = h_G(x_G, u, y)
\end{cases} \quad (23)$$

$$R_K : \begin{cases}
\dot{x}_K = f_K(x_K, y, u) \\
\dot{z}_K = h_K(x_K, y, u)
\end{cases} \quad (24)$$

Then all the strongly internally stable plant-controller pairs $G_S$ and $Q_K$ are given by

$$G_S : \begin{cases}
\dot{x}_S = f_S(x_K, u, h_{G_S}(x_{GSK}, x_S, u)) \\
\dot{z}_S = h_{G_S}(x_{GSK}, x_S, u) \\
y = h_{G_S}(x_{GSK}, x_S, u)
\end{cases} \quad (25)$$

$$Q_K : \begin{cases}
\dot{x}_Q = f_Q(x_K, y, h_{Q_K}(x_{GSK}, x_Q, y)) \\
\dot{z}_Q = h_{Q_K}(x_{GSK}, x_Q, y) \\
u = h_{Q_K}(x_{GSK}, x_Q, y)
\end{cases} \quad (26)$$

where a strongly internally stable system $(S, Q)$ is defined by the following equations

$$S : \begin{cases}
\dot{x}_S = f_S(x_S, z_K) \\
z_G = h_S(x_S, z_K)
\end{cases} \quad (27)$$

$$Q : \begin{cases}
\dot{x}_Q = f_Q(x_Q, z_S) \\
z_K = h_Q(x_Q, z_S)
\end{cases} \quad (28)$$

such that the equations of the output functions $h_{G_S}, h_{K}, h_{S}$ and $h_{G_K}, h_{Q_K}, h_{Q}$ can be solved for $y, u$ respectively as follows and moreover that the equations (29) and (30) have a unique solution $(u, y)$:

$$y = h_{G_S}(x_{GSK}, x_S, u) \quad (29)$$

$$u = h_{Q_K}(x_{GSK}, x_Q, y) \quad (30)$$

Furthermore, the initial states $x_0^G, x_0^K, x_0^G, x_0^K, x_0^S$ and $x_0^Q$ are independent.

Theorem 3 gives the parametrization of the pairs of strongly internally stable $G_S$ and $Q_K$ in the operator setting which is also valid in state space setting. In Corollary 1, the independence of the initial states of the operators allows us to connect all the operators only by their input and output (c.f. see section 5).

In these realizations, the state $x_K$ in $G_S$ and the state $x_G$ in $Q_K$ can be regarded as the estimated state of $x_K$ in $G_Q$ and that of $x_G$ in $G_S$ respectively. The following section discusses the construction of the detectable kernel representation based on a kind of state observer.

### 3.4 State-space realization of a detectable kernel representation

We will show the example of constructing the realization of a strongly detectable kernel representation $R_{GK}$ by a state observer and a state feedback for the plant $G$. Now $G$ is assumed to have the following realization.

$$G : \begin{cases}
\dot{x} = f_G(x, u, y) \\
y = h(x)
\end{cases} \quad (31)$$

Suppose the existence of a state observer and a state feedback of the form

$$\dot{x} = f_G(x, u, y) \quad (32)$$

$$u = k(x) \quad (33)$$

and construct the following operator $T_{GK}$.

$$T_{GK} : \begin{cases}
\dot{\hat{u}} = f_G(x, u, y) \\
\dot{\hat{y}} = h(x)
\end{cases} \quad (34)$$

We make some assumptions for $T_{GK}$.

- **(A1) Stability of observer**: $T_{GK} : (u, y) \mapsto (\hat{u}, \hat{y})$ is stable.
- **(A2) Stability of closed-loop**: The resulting system by adding the following uniform feedback to $T_{GK}$: $e \mapsto (\hat{u}, \hat{y})$ is stable. Moreover, $(\hat{u}, \hat{y})$ is stable if and only if $e$ is stable.

$$u = h(x) \quad (35)$$

- **(A3) Detectability of observer**: Consider two operators $T_{GK}^1$ and $T_{GK}^2$, where $x^1$ and $x^2$ are independent, and connect them as follows. Then $e$ is stable if and only if $(e_1, e_2)$ is stable.

$$\dot{x}_1 = f_G(x_1, u, y) \quad (36)$$

$$\dot{x}_2 = f_G(x_2, u + e_1, y + e_2) \quad (37)$$

$$\dot{e} = \left( \begin{array}{c} k(x_1) - k(x_2) \\ h(x_1) - h(x_2) \end{array} \right) \quad (38)$$

The assumption A1 means the stability of the state observer (32), and A2 means that the pair of observer and feedback stabilizes the plant. A3 expresses the strong detectability of the observer, i.e. for any stable disturbance $(e_1, e_2)$, the two observers (36) and (37) estimate so close $x_1$ and $x_2$ that the difference signal (38) is stable.

**Proposition 1**: Consider a plant $G$ in the form (31) such that there exist a state observer (32) and the stabilizing state feedback (33) for $G$ satisfying the assumptions A1, A2 and A3. Then $R_{GK}$ defined by the following is strongly internally stable and strongly detectable.

$$R_G : \begin{cases}
\dot{x}_G = f_G(x_G, u, y) \\
z_G = h(x_G)
\end{cases} \quad (39)$$

$$R_K : \begin{cases}
\dot{x}_K = f_G(x_K, u, y) \\
z_K = u - k(x_K)
\end{cases} \quad (40)$$
Proof is straightforward by checking conditions. If we choose the kernel representation \( R_{GK} \) as in Proposition 1, then the parametrized controller \( K_{Q} \) in Corollary 1 (Theorem 3) is depicted in Figure 2. This is quite similar to the linear Youla-Kucera parametrization and is a natural nonlinear extension. Also, \( R_{Gz} \) can be realized in a similar form in this case.

\[ \begin{align*}
  u' & \rightarrow x_K = f_G(x_K, u', y) \\
  & \quad \downarrow k(x_K) \quad h(x_K) \\
  & \quad \downarrow Q \\
  & \quad z_K \\
  & \quad \downarrow z_G
\end{align*} \]

**Figure 2:** Parametrization of \( K_Q \) using state observer and state feedback

Proposition 1 suggests that detectable kernel representations in state-space realizations will be realized using state observers. The assumptions A3 is very strong assumptions, and Proposition 1 will be applicable for general nonlinear systems by modifying it into local setting. It should be noticed that Proposition 1 (or Corollary 1 directly) is easily applicable for linear systems and that, if the nominal plant is linear, then this parametrization gives all the nonlinear stabilizing plant-controller pairs.

### 4 Youla-Kucera parametrization

This section discusses the parametrization of stabilizing controllers for a fixed plant, namely, Youla-Kucera parametrization. Of course, we can get the parametrization of stabilizing controllers for a fixed \( G \) when setting \( S = 0 \) (\( R_S \) is not fixed in this case) in Theorem 2 or 3. Moreover, the parametrization of stabilizing controllers for a fixed kernel representation \( R_G \) can be given when setting \( S = 0 \) and \( R_S(z_K, z_G) = z_G \) in Theorem 2 or 3. In the latter case, however, the detectability of \( R_{GK} \) is a strong assumption for the parametrization since \( R_K \) is not used in it, and actually the assumption can be relaxed into the detectability only of kernel representation \( R_G \). This section considers the parametrization in the latter case, namely the parametrization of all the stabilizing controllers for a fixed kernel representation \( R_G \).

**Definition 7:** A kernel representation \( R_G \) of a system \( G \) is detectable if the following holds for \( \forall (x_G^0, x_G^1, u, y) \in X_G^0 \times X_G^1 \times W \):

\[ R_G(u, y) \in Z_G^0 \iff R_G(u, y) \in Z_G^0 \]  

Also \( R_G \) is strongly detectable if the following holds for \( \forall (x_G^0, x_G^1, u, y, e_1, e_2) \in X_G^0 \times X_G^1 \times W \times \mathcal{E}_U^2 \):

\[ R_G(u, y) \in Z_G^0 \iff R_G(u + e_1, y + e_2) \in Z_G^0 \]  

**Figure 3** explains the concept of the detectability of \( R_G \). The detectability of \( R_G \) is equivalent to the stability of the mapping \( z_G \mapsto z_G \) for \( \forall u \in U \). This means \( R_G \) is an observer or an estimator of \( G_{zG} \) as in the previous section. Assuming the detectability of \( R_G \), the following result is derived.

**Theorem 4:** Consider a system \( \{G, K\} \) with a kernel representation \( R_{GK} \) and suppose it is (strongly) internally stable and \( R_G \) is (resp. strongly) detectable. Then all the controller \( K^* \) such that the system \( \{G, K^*\} \) with a kernel representation \( R_{GK^*} \), where \( R_{K^*} : Y \times U \rightarrow Z_{K^*} \), is (resp. strongly) internally stable are given by the equation (18) where \( R_Q \) is a well-defined stable kernel representation of a stable system \( Q \), i.e. \( R_Q : Z_G \times Z_K \rightarrow Z_Q = Z_{K^*} \), such that \( R_Q = 0 \) is stable and that \( R_{GK_Q} \) is (resp. strongly) well-posed. Furthermore, the initial states \( x_G^0, z_K, z_G^0 \) and \( x_G^0 \) are independent.

Proof is similar to that of Theorem 3 and is omitted for reasons of space. Theorem 4 shows the parametrization of all the stabilizing controller \( K_Q \) for a fixed plant \( G_{z_G} \). It should be noted that this parametrization only needs the detectability of \( R_G \), which is much weaker than that of \( R_{GK} \). Theorem 4 can be expressed by state space realizations.

**Corollary 2:** Consider a system \( \{G, K\} \) with the kernel representation \( R_{GK} \) as in (23) and (24), and suppose it is (strongly) internally stable and \( R_G \) is (resp. strongly) detectable. Then all the (resp. strongly) internally stabilizing \( K_Q \) is given by the equation (26) where a stable operator \( \bar{Q} \) is defined by the equation (28) such that the equations of the output functions \( h_G, h_K, h_Q \) can be solved for \( u \) as in (30) and moreover that the equations (30) and substituting \( z_G = 0 \) (resp. substituting \( (u, y, z_G) = (u + e_1, y + e_2, 0) \)) for (23) have a unique solution \( (u, y) \). Furthermore, the initial states \( x_G^0, z_K, x_G^0, z_G^0 \) and \( x_G^0 \) are independent.

### 5 Relationship to state-space results

This section considers the relationship between the existing state-space approach results [2]–[4] and the kernel representation approach taken in the present paper. The state-space approach is based on a weaker concept of stability than that used in the previous sections, that is the stability when \( z_G = 0 \). This stability is stated as follows.

**Definition 8:** A closed loop system \( \{G, K\} \) with a kernel representation \( R_{GK} \) is weakly stable if \( R_{GK} \) is invertible for \( \forall x_G^0 \in X_G^0 \) and the system \( \{G, K_{zk} \} \) is stable\(^1\) if and only if \( z_K \in Z_K \) for \( \forall x_G^0 \in X_G^0 \), i.e.

\[ z_K \in Z_K \iff R_{GK}^{-1}(z_K, 0) \in W^s \]  

where \( x_G^0 \) and \( z_K^0 \) are independent.

\( ^1 \)When state-space realizations are considered, the state \( x_G^0 \) is also stable.

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p. 5
Moreover, weak detectability is defined corresponding to weak stability.

**Definition 9**: A kernel representation $R_G$ of a system $G$ is weakly detectable if the following holds for \( \forall (x_G^0, x_G^1, u) \in X_G^0 \times X_G^1 \times U \):

\[
R_G(u, G(u)) \in Z_G^0
\]  

(44)

Since weakly stable $R_{GK}$ is not necessarily unimodular (coprime), the similar parametrization as in Theorem 4 does not give all the stabilizing controllers.

**Theorem 5**: Consider a system $\{G, K\}$ with a kernel representation $R_{GK}$ and suppose it is weakly stable and $R_G$ is weakly detectable. Then the system $\{G, K_Q\}$ with a kernel representation $R_{GK_Q}$ defined as in (15) is weakly stable, where $R_Q$ is a stable kernel representation of a stable system $Q$, i.e. $R_Q : Z_G \times Z_K \to Z_Q$, such that $R^a_Q$ is stable and that $\{G, K_Q\}$ with $R_{GK_Q}$ is well-posed.

Proof is clear and is omitted. Though Theorem 5 gives the parametrization of a class of globally stabilizing controllers, it does not give all the stabilizing ones. However, if the signal stability is specified as in the following proposition, the parametrization in Theorem 5 gives all the locally stabilizing controllers. We have to define some more notations before stating the proposition. A signal $z$ is said to be a converging signal if there exists a class-$L$ function $\alpha(\cdot)$ satisfying $|x(t)| \leq \alpha(t)$. A scalar function $\alpha(\cdot)$ is said to be a class-$L$ function if it is monotonously decreasing and $\lim_{t \to \infty} \alpha(t) = 0$ holds. A scalar function $\alpha(\cdot)$ is said to be a class-$K$ function if it is monotonously increasing and $\alpha(0) = 0$ holds. Moreover, Input-to-state stability means that the state is converging if the input signal is converging [9]–[11].

**Proposition 2**: Define the set of stable signals $Z^s$ to denote the set of converging signals in $Z$ and the stability of the state to be input-to-state stability\(^2\). Given a weakly stable system $\{G, K^s\}$ with a kernel representation $R_{GK}$, such that $R_G$ is weakly detectable where $R_{GK} : Y \times U \to Z_K$, then there exists a stable kernel representation $R_{QK} : Z_G \times Z_K \to Z_Q$, such that $K_Q = K^s$ and $R^Q_{QK}$ is locally stable (with small input).

**Proof**: We only prove the local stability of the operator $Q$. Suppose (28) denotes the state-space realization of $Q$, then the signal $x_Q$ derived from the following system is converging.

\[
\dot{x}_Q = f_Q(x_Q, 0)
\]  

(45)

This implies, by the converse Lyapunov theorem [15], that there exists a Lyapunov function $V_Q(x_Q)$ satisfying

\[
\alpha_1(|x_Q|) \leq V_Q(x_Q) \leq \alpha_2(|x_Q|)
\]  

(46)

\[
\dot{V}_Q(x_Q) \leq -\alpha_3(|x_Q|) + \alpha_4(|z_Q|)
\]  

(47)

where $\alpha_i(\cdot)$'s are class-$K$ functions. This implies $Q$ is locally stable (with small input).

Unfortunately, Theorem 5 is not valid in the case the initial states of all the operators are independent, i.e. in this parametrization the initial state of $Q$ depends on that of $R_{GK}$. If we reformulate the operator can always know the state of $R_{GK}$, the initial states can be independent. In such case, however, the parametrization becomes more complicated and it is depicted in Figure 4. The figure $Q : (z_G, z_K) \mapsto z_K$ is a parameter operator such that $z_K$ is stable if $z_G$ is stable for any $x_{GK}$.

**Figure 4**: Parametrization of weakly stabilizing $K_Q$

Proposition 2 implies the local version of such a parametrization with appropriate setting gives all the locally stabilizing controllers. This result is equivalent to Imura et. al. [2]. We do not give the details of this result for reasons of space, but this shows the consistency between the kernel representation approach and the existing state space approach.

**References**


